International Journal of Theoretical Physics, Vol. 31, No. 8, 1992

Necessary and Sufficient Conditions for Existence of Exponential Dichotomy of Impulsive Equations

D. D. Bainov,[1] S. I. Kostadinov,[1] and P. P. Zabreiko[2]

Received May 4, 1991

Necessary and sufficient conditions for the existence of an exponential dichotomy of impulsive differential equations in a Hilbert space are found.

1. INTRODUCTION

In the present paper we introduce the notion of *generalized exponential dichotomy* for linear impulsive differential equations. Necessary and sufficient conditions for its existence are found. Similar problems for equations without impulse effect and in finite-dimensional spaces have been studied in Coppel (1978), Muldowney (1984), and Sacker and Sell (1974).

2. STATEMENT OF THE PROBLEM

Let X be a Hilbert space with scalar product $(\cdot\ \cdot\ \cdot)$ and let I be the identical operator in X, and $L(X)$ be the space of linear bounded operators acting in X. Consider the impulsive differential equation

$$\frac{dx}{dt} = A(t)x \qquad (t \neq t_n) \tag{1}$$

$$x(t_n^+) = Q_n x(t_n) \tag{2}$$

where the function A: $R_+ = [0, \infty) \to L(X)$ is continuous on each interval $[t_n, t_{n+1}]$. We have $Q_n \in L(X)$, and the inverse bounded operators $Q_n^{-1} \in L(X)$ exist. The points t_n satisfy the condition

$$t_n < t_{n+1} \quad (n \in N), \qquad \lim_{n\to\infty} t_n = \infty$$

[1]Plovdiv University, Plovdiv, Bulgaria.
[2]Byelorussian State University, Minsk, Byelorussia.

0020-7748/92/0800-1521$06.50/0

Let $U(t)$ $(t \geq 0)$ (Bainov and Kostadinov, 1987; Bainov *et al.*, 1989; Zabreiko *et al.*, 1987, 1988) be the Cauchy operator of equation (1), (2) for which $U(0) = I$.

Definition 1. The linear impulsive differential equation (1), (2) is said to be *generalized exponentially dichotomous* if there exist positive functions $\mu_1(t)$ and $\mu_2(t)$, a constant M, and projectors P_1 and P_2 $(P_1 + P_2 = I)$ for which the following inequalities are valid:

$$\| W_1(t, s) \| \leq M \exp\left[-\int_s^t \mu_1(\tau)\, d\tau \right] \qquad (0 \leq s \leq t < \infty) \tag{3}$$

$$\| W_2(t, s) \| \leq M \exp\left[-\int_t^s \mu_2(\tau)\, d\tau \right] \qquad (0 \leq t \leq s < \infty) \tag{4}$$

where $W_i(t, s) = U(t) P_i U^{-1}(s)$ $(i = 1, 2)$.

If the functions $\mu_1(t)$ and $\mu_2(t)$ are integrally bounded, i.e., there exists a constant $\mu > 0$ for which

$$\int_a^b \mu_i(\tau)\, d\tau \leq \mu(b - a) \qquad (a \leq b; \quad i = 1, 2)$$

then the impulsive equation (1), (2) is said to be *exponentially dichotomous.* If $\mu = 0$, then equation (1), (2) is said to be *dichotomous.*

3. MAIN RESULTS

Theorem 1. Let the impulsive equation (1), (2) be generalized exponentially dichotomous and let the functions $\mu_i(t)$, $i = 1, 2$, be uniformly positive: $\mu_i(t) \geq \varepsilon_0 > 0$ $(t \in R_+)$.

Then there exists an operator-valued function $H(t)$ $(0 \leq t < \infty)$ with the following properties:

1. $H(t)$ is continuously differentiable for $t \neq t_n$, has points of discontinuity of the first kind at $t = t_n$, and is bounded.
2. $H(t)$ is a self-adjoint operator for each $t \in R_+$.
3. $(C(t)z, z) \leq -\eta \|z\|^2$ $(z \in X)$, where

$$C(t) = H'(t) + H(t)A(t) + A^*(t)H(t)$$

and $\eta > 0$ is a constant.

Proof. Define the operator-valued function $H(t)$ by the formula

$$H(t) = 2\left\{ \int_t^\infty W_1^*(s, t) W_1(s, t)\, ds - W_2^*(0, t) W_2(0, t) - \int_0^t W_2^*(s, t) W_2(s, t)\, ds \right\}$$

We shall show the boundedness of $H(t)$:

$$\|H(t)\| \le 2\left\{\int_t^\infty \|W_1(s,t)\|^2\,ds + \|W_2(0,t)\|^2 + \int_0^t \|W_2(s,t)\|^2\,ds\right\}$$

$$\le 2\left\{M^2\int_t^\infty \exp\left[-2\int_t^s \mu_1(\tau)\,d\tau\right]ds + M^2\exp\left[-2\int_0^t \mu_2(\tau)\,d\tau\right]\right.$$

$$\left. + M^2\int_0^t \exp\left[-2\int_s^t \mu_2(\tau)\,d\tau\right]ds\right\}$$

$$\le 2M^2\left(\frac{1}{2\varepsilon_0}+1+\frac{1}{2\varepsilon_0}\right) = 2M^2\left(\frac{1}{\varepsilon_0}+1\right) < \infty$$

The proof of the remaining assertions of Theorem 1 is a modification of the proof of Theorem 1 of Coppel (1978, p. 59). ■

Corollary 1. Let the conditions of Theorem 1 hold and let the function $A(t)$ be bounded. Then the derivative $H'(t)$ is bounded.

The *proof* of Corollary 1 follows from the representation

$$H'(t) = -H(t)A(t) - A^*(t)H(t) - 2W_1^*(t,t)W_1(t,t) \qquad (t \ne t_n)$$

Definition 2. The impulsive equation (1), (2) is said to be of *bounded growth* if for some fixed $h > 0$ there exists a constant $C \ge 1$ such that each solution $x(t)$ satisfies the condition

$$\|x(t)\| \le C\|x(s)\| \qquad \text{for} \quad a \le t \le s+h$$

Theorem 2. Let the following conditions hold:

1. Equation (1), (2) is of bounded growth.

2. $\dim X_2 < \infty$, where X_2 is some complement to the space X_1 which consists of all $\xi \in X$ for which the impulsive equation (1), (2) has a bounded solution $x(t)$ on R_+ for which $x(0) = \xi$.

3. There exists a function $V(t,x)$: $R_+ \times X \to R$ which enjoys the following properties:

(i) For fixed t the function $V(t,x)$ for $x \ne 0$ is differentiable with respect to x and for fixed x it is continuously differentiable with respect to t for $t \ne t_n$ and has discontinuities of the first kind at the points $t = t_n$.

(ii) There exist constants ρ, $\sigma > 0$ such that

$$|V(t,x)| \le \rho\|x\|^2 \qquad (t \in R_+, x \in X)$$

and for each solution $x(t)$ of equation (1), (2) the following inequality is valid:

$$V(t,x(t)) - V(\tau,x(\tau)) \le -\sigma\int_\tau^t \|x(u)\|^2\,du \qquad (0 \le \tau \le t < \infty)$$

(iii) For an arbitrary solution $x(t)$ of equation (1), (2) with initial condition $\xi \in X_2$ the following equality is valid:

$$\lim_{t\to\infty} V(t, x(t)) = -\infty$$

Then equation (1), (2) is exponentially dichotomous.

Proof. Let $x(t)$ be an arbitrary solution of (1), (2). We first consider the case when the function $V(t, x(t))$ is nonnegative. Then from the inequalities

$$\int_0^t \|x(u)\|^2 \, du \le \sigma^{-1}[V(0, x(0)) - V(t, x(t))] \le \sigma^{-1} V(0, x(0))$$

it follows that

$$\int_0^\infty \|x(u)\|^2 \, du < \infty$$

and therefore

$$\underline{\lim}_{t\to\infty} \|x(t)\|^2 = 0$$

Denote by τ_m, $m = 1, 2, 3, \ldots$, the least of the numbers for which $\|x(\tau_m^+)\| \le e^{-m/2}$. Then $0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 < \cdots$, and by condition (ii) of Theorem 2 for $t > \tau_m$ we obtain the inequalities

$$-\rho\|x(t)\|^2 \le -V(t, x(t)) \le V(\tau_{m+1}, x(\tau_{m+1})) - V(t, x(t))$$

$$\le -\sigma \int_t^{\tau_{m+1}} \|x(u)\|^2 \, du \le -\sigma(\tau_{m+1} - t)\, e^{-(m+1)}$$

which imply the inequality $\tau_{m+1} - \tau_m \le e\rho\sigma^{-1}$.

Let $0 \le s \le t < \infty$ and let m and n be numbers such that $\tau_m \le s < \tau_{m+1}$ and $\tau_n \le t < \tau_{n+1}$ $(0 \le m \le n < \infty)$. Since the impulsive equation (1), (2) is of bounded growth, then from the definition of the numbers τ_n it follows that there exists $\tilde{t} > \tau_n$ for which

$$\|x(t)\| \le C\|x(\tilde{t})\| \le eC^{-(n+1-m)/2}\|x(s)\|$$

whence it follows that

$$\|x(t)\| \le eC\, e^{-\alpha(t-s)}\|x(s)\|$$

where $\alpha = (2e\rho\sigma^{-1})^{-1}$.

The last inequality means that all solutions of equation (1), (2) exponentially tend to zero, and hence equation (1), (2) is exponentially dichotomous.

Now let the function $V(t, x(t))$ be negative at the point t_*. Then for $t > t_*$ from condition (ii) it follows that the function $V(t, x(t))$ will be negative and

$$V(t, x(t)) - V(t_*, x(t_*)) \le -\sigma \int_{t_*}^{t} \|x(u)\|^2 \, du \le \frac{\sigma}{\rho} \int_{t_*}^{t} V(u, x(u)) \, du$$

From Gronwall's lemma we obtain

$$V(t, x(t)) \le V(t_*, x(t_*)) \, e^{(\sigma/\rho)(t - t_*)}$$

From the last inequality it follows that

$$\lim_{t \to \infty} V(t, x(t)) = -\infty \tag{5}$$

with which we proved that the solution $x(t)$ is unbounded.

We shall show that the limit in (5) is uniform for all solutions $x(t)$ of (1), (2) with initial values $x(0)$ belonging to the unit sphere of the space X_2. Indeed, otherwise there would exist a sequence $u_m \to \infty$ and initial conditions ξ_m for which for some μ the corresponding solution $x_m(t)$ should satisfy the condition

$$V(u_m, x_m(u_m)) \ge \mu \tag{6}$$

Without loss of generality we can assume that the sequence of solutions $\{x_m(t)\}$ tends to the solution $x(t)$ of equation (1), (2) with initial value ξ. Since for this solution, equality (5) is valid, then for some $\tilde{t}$

$$V(\tilde{t}, x(\tilde{t})) < \mu$$

That is why for large values of m the following inequality is valid:

$$V(\tilde{t}, x_m(\tilde{t})) < \mu$$

as well as, for $t > t_m$, the inequality

$$V(t, x_m(t)) < \mu$$

The last inequality contradicts (6), since $u_m \to \infty$. Hence

$$\lim_{t \to \infty} \sup_{\xi \in X_2, \|\xi\| = 1} V(t, x_\xi(t)) = -\infty$$

where $x_\xi(t)$ is a solution of (1), (2) with initial condition ξ. From condition (ii) of Theorem 2 it follows that

$$\lim_{t \to \infty} \inf_{\xi \in X_2, \|\xi\| = 1} \|x_\xi(t)\| = \infty$$

Hence there exists a number $T > 0$ such that for any $\xi \in X_2$ and $\|\xi\| = 1$ and for all $t > T$ the following inequality is valid:

$$\|x_\xi(t)\| > 1$$

Moreover, a number $N \geq 1$ can be chosen such that $\|x_\xi(T)\| \leq N\|x_\xi(t)\|$ for each $\xi \in X_2$, $\|\xi\| = 1$, and $0 \leq t \leq T$.

Consider again the solution $x(t) = x_\xi(t)$. Since $\|x_\xi(t)\| \to_{t\to\infty} \infty$, then there exists a maximal τ_m for which $\|x(\tau_m)\| \leq e^{m/2}$. Let $t \geq T$. For $\tau_m \leq t < \tau_{m+1}$ we have

$$\begin{aligned} -\rho\|x(\tau_{m+1})\|^2 &\leq V(\tau_{m+1}, x(\tau_{m+1})) \\ &\leq V(\tau_{m+1}, x(\tau_{m+1})) - V(t, x(t)) \\ &\leq -\sigma \int_t^{\tau_{m+1}} \|x(u)\|^2\, du \\ &\leq -\sigma(\tau_{m+1} - t)\|x(t)\|^2 \end{aligned} \tag{7}$$

From inequalities (7) it follows that $\tau_{m+1} - \tau_m \leq e\rho\sigma^{-1}$ for $\tau_m \geq T$ and $\tau_{m+1} - T \leq e\rho\sigma^{-1}$ for $\tau_m < T$. In both cases we have

$$\|x(t)\| \leq CN\|x(\tau_m)\|$$

Let $T \leq t \leq s < \infty$. For $\tau_m \leq t < \tau_{m+1}$ and $\tau_n \leq s < \tau_{n+1}$ $(0 \leq m \leq n)$ we have

$$\begin{aligned} \|x(t)\| &\leq CN\|x(\tau_m)\| \\ &\leq e^{1/2}CN\,e^{-(n+1-m)/2}\|x(\tau_n)\| \\ &\leq e^{1/2}CN\,e^{-(n+1-m)/2}\|x(s)\| \\ &\leq e^{1/2}CN\,e^{-\alpha(s-t)}\|x(s)\| \end{aligned}$$

where $\alpha = (2e\rho\sigma^{-1})^{-1}$. Since equation (1), (2) is of bounded growth, then it is exponentially dichotomous on the interval $[T, \infty)$, that is, on the half-axis R_+, too. ■

ACKNOWLEDGMENT

This investigation was supported by the Bulgarian Ministry of Science and Higher Education under Grant 61.

REFERENCES

Bainov, D. D., and Kostadinov, S. I. (1987). *Collectanea Mathematica*, **37**(3), 193–198.

Bainov, D. D., Kostadinov, S. I., and Zabreiko, P. P. (1989). *Mathematical Reports of Toyama University*, **12**, 159–166.

Coppel, W. A. (1978). *Dichotomies in Stability Theory*, Springer-Verlag, Berlin, p. 97.

Muldowney, J. (1984). *Transactions of the American Mathematical Society*, **283**(2), 465–484.

Sacker, R. J., and Sell, G. R. (1974). *Journal of Differential Equations*, **15**, 429–458.

Zabreiko, P. P., Bainov, D. D., and Kostadinov, S. I. (1987). *Tamkang Journal of Mathematics*, **18**(4), 57–63.

Zabreiko, P. P., Bainov, D. D., and Kostadinov, S. I. (1988). *International Journal of Theoretical Physics*, **27**(6), 731–743.